SELECTIONS, EXTENSIONS AND COLLECTIONWISE NORMALITY

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ABSTRACT. We demonstrate that the classical Michael selection theorem for l.s.c. mappings with a collectionwise normal domain can be reduced only to compact-valued mappings modulo Dowker's extension theorem for such spaces. The idea used to achieve this reduction is also applied to get a simple direct proof of that selection theorem of Michael's. Some other possible applications are demonstrated as well.

1. INTRODUCTION

For a topological space E, let 2^E be the family of all nonempty subsets of E; $\mathscr{F}(E)$ — the subfamily of 2^E consisting of all closed members of 2^E; and let $\mathscr{C}(E)$ be that one of all compact members of $\mathscr{F}(E)$. Also, let $\mathscr{C}'(E) = \mathscr{C}(E) \cup \{E\}$. A set-valued mapping $\varphi : X \to 2^E$ is *lower semi-continuous*, or l.s.c., if the set

$$\varphi^{-1}(U) = \{x \in X : \varphi(x) \cap U \neq \varnothing\}$$

is open in X for every open $U \subset E$. A map $f : X \to E$ is a *selection* for $\varphi : X \to 2^E$ if $f(x) \in \varphi(x)$ for every $x \in X$.

Recall that a space X is *collectionwise normal* if it is a T_1-space and for every discrete collection $\mathscr{D}$ of closed subsets of X there exists an open discrete family $\{U_D : D \in \mathscr{D}\}$ such that $D \subset U_D$ for every $D \in \mathscr{D}$. Every collectionwise normal space is normal, but the converse is not necessarily true [1], see, also, [7, 5.1.23 Bing's Example]. It is well known that a T_1-space X is collectionwise normal if and only if for every closed subset $A \subset X$, every continuous map from A to a Banach space E can be continuously extended to the whole of X, Dowker [5]. Generalizing this result, Michael [16] stated the following theorem.

Theorem 1.1 ([16]). *For a T_1-space X, the following are equivalent*:
  (a) *X is collectionwise normal.*
  (b) *If E is a Banach space and $\varphi : X \to \mathscr{C}'(E)$ is an l.s.c. convex-valued mapping, then φ has a continuous selection.*

However, the arguments in [16] for (a) $\Rightarrow$ (b) of Theorem 1.1 were incomplete and, in fact, working only for the case of $\mathscr{C}(E)$-valued mappings. The first complete proof of this implication was given by Choban and Valov [2] using a different technique. We are now ready to state also the main purpose of this paper. Namely, in this paper we prove the following theorem which demonstrates that the original Michael arguments in [16] have been actually adequate to the proof of Theorem 1.1.

Theorem 1.2. *For a Banach space E, the following are equivalent*:
  (a) *If X is a collectionwise normal space and $\varphi : X \to \mathscr{C}(E)$ is an l.s.c. convex-valued mapping, then φ has a continuous selection.*
  (b) *If X is a collectionwise normal space and $\varphi : X \to \mathscr{C}'(E)$ is an l.s.c. convex-valued mapping, then φ has a continuous selection.*

Let us emphasize that the proof of Theorem 1.2 is based only on Dowker's extension theorem [5]. This proof is presented in the next section and its main ingredient is the fact that if $\varphi : X \to \mathscr{C}'(E)$ and $g : X \to E$, then $\varphi(x)$ is compact for every $x \in X$ for which $g(x) \notin \varphi(x)$. This is further applied in Section 3 to get with ease a direct proof of a natural generalization of Theorem 1.1. Section 4 deals with controlled selections for set-valued mappings defined on countably paracompact or collectionwise normal spaces which are naturally interrelated to the idea of Theorem 1.2.

2. Proof of Theorem 1.2

It only suffices to prove that (a) $\Rightarrow$ (b). To this end, suppose that (a) of Theorem 1.2 holds, E is a Banach space, and X is a collectionwise normal space. Here, and in the sequel, we will use d to denote the metric on E generated by the norm of E. Recall that a map $f : X \to E$ is an ε-*selection* for $\psi : X \to 2^E$ if $d(f(x), \psi(x)) < \varepsilon$ for every $x \in X$.

The key element in the proof of this implication is the following construction of approximate selections.

Claim 2.1. *Let $\psi : X \to 2^E$ be an l.s.c. convex-valued mapping and $g : X \to E$ be a continuous map such that $\psi(x)$ is compact whenever $x \in X$ and $g(x) \notin \psi(x)$. Then, for every $\varepsilon > 0$, ψ has a continuous ε-selection.*

Proof. Let $\varepsilon > 0$ and $A = \left\{ x \in X : d(g(x), \psi(x)) \geq \varepsilon \right\}$. Then, $A \subset X$ is closed because ψ is l.s.c. and g is continuous. Since $\psi \restriction A : A \to \mathscr{C}(E)$ and A is itself a collectionwise normal space, by (a) of Theorem 1.2, $\psi \restriction A$ has a continuous selection $h_0 : A \to E$. Since X is collectionwise normal, by Dowker's extension theorem [5], there exists a continuous map $h : X \to E$ such that $h \restriction A = h_0$. Consider the set $U = \left\{ x \in X : d(h(x), \psi(x)) < \varepsilon \right\}$ which contains A and is open because ψ is l.s.c. and h is continuous. Finally, take a continuous function

$\alpha : X \to [0,1]$ such that $A \subset \alpha^{-1}(0)$ and $X \setminus U \subset \alpha^{-1}(1)$, and then define a continuous map $f : X \to E$ by

$$f(x) = \alpha(x) \cdot g(x) + \big(1 - \alpha(x)\big) \cdot h(x), \quad x \in X.$$

This f is as required. Indeed, take a point $x \in X$. If $x \in A$, then $\alpha(x) = 0$ and, therefore, $f(x) = h(x) = h_0(x) \in \psi(x)$. If $x \in X \setminus U$, then $\alpha(x) = 1$ and we now have that $f(x) = g(x)$, so $d(f(x), \psi(x)) = d(g(x), \psi(x)) < \varepsilon$ because $x \notin A$. Suppose finally that $x \in U \setminus A$. In this case, $d(h(x), \psi(x)) < \varepsilon$ and $d(g(x), \psi(x)) < \varepsilon$. Since $\psi(x)$ is convex and $f(x) = \alpha(x) \cdot g(x) + \big(1 - \alpha(x)\big) \cdot h(x)$, this implies that $d(f(x), \psi(x)) < \varepsilon$. The proof is completed. $\square$

Having already established Claim 2.1, we proceed to the proof of (a) $\Rightarrow$ (b) which is based on standard arguments for constructing continuous selections, see [16]. In this proof, and in what follows, for a nonempty subset $S \subset E$ and $\varepsilon > 0$, we will use $B_\varepsilon(S) = \{y \in E : d(y, S) < \varepsilon\}$ to denote the *open ε-neighbourhood* of S. In particular, for a point $y \in E$, we set $B_\varepsilon(y) = B_\varepsilon(\{y\})$.

Let $\varphi : X \to \mathscr{C}'(E)$ be an l.s.c. convex-valued mapping. If $g : X \to E$ is any continuous map, say a constant one, then $\varphi(x)$ is compact for every $x \in X$ for which $g(x) \notin \varphi(x)$. Hence, by Claim 2.1, φ has a continuous 2^{-1}-selection $f_0 : X \to E$. Define $\varphi_1 : X \to \mathscr{F}(E)$ by

$$\varphi_1(x) = \overline{\varphi(x) \cap B_{2^{-1}}(f_0(x))}, \quad x \in X.$$

According to [16, Propositions 2.3 and 2.5], φ_1 is l.s.c., and clearly it is convex-valued. Finally, observe that if $f_0(x) \notin \varphi_1(x)$ for some $x \in X$, then $f_0(x) \notin \varphi(x)$ and, therefore, $\varphi(x)$ is compact because it is $\mathscr{C}'(E)$-valued. Since $\varphi_1(x)$ is a closed subset of $\varphi(x)$, it is also compact. Hence, by Claim 2.1, φ_1 has a continuous 2^{-2}-selection f_1. In particular, f_1 is a continuous 2^{-2}-selection for φ such that

$$d(f_1(x), f_0(x)) \le 2^{-1} < 2^0, \quad \text{for every } x \in X.$$

Thus, by induction, we get a sequence $\{f_n : n < \omega\}$ of continuous maps such that, for every $n < \omega$ and $x \in X$,

(2.1) $d(f_n(x), \varphi(x)) < 2^{-(n+1)}$,

(2.2) $d(f_{n+1}(x), f_n(x)) < 2^{-n}$.

By (2.2), $\{f_n : n < \omega\}$ is a Cauchy sequence in the complete metric space (E, d), so it must converge to some continuous $f : X \to E$. By (2.1), $f(x) \in \varphi(x)$ for every $x \in X$. Hence, (b) holds and the proof of Theorem 1.2 is completed.

3. More on Selections and Collectionwise Normality

A space X is *τ-collectionwise normal*, where τ is an infinite cardinal number, if it is a T_1-space and for every discrete collection $\mathscr{D}$ of closed subsets of X, with $|\mathscr{D}| \le \tau$, there exists a discrete collection $\{U_D : D \in \mathscr{D}\}$ of open subsets of X

such that $D \subset U_D$ for every $D \in \mathscr{D}$. Clearly, a space X is collectionwise normal if and only if it is τ-collectionwise normal for every τ. Also, it is well known that X is normal if and only if it is ω-collectionwise normal. However, for every τ there exists a τ-collectionwise normal space which is not τ^+-collectionwise normal [19], where τ^+ is the immediate successor of τ.

The proof of Theorem 1.2 suggests an easy direct proof of the following natural generalization of Theorem 1.1 in [2].

Theorem 3.1 ([2]). *Let X be a τ-collectionwise normal space, E be a Banach space with a topological weight $w(E) \leq \tau$, and let $\varphi : X \to \mathscr{C}'(E)$ be an l.s.c. convex-valued mapping. Then, φ has a continuous selection.*

Proof. It only suffices to prove the statement of Claim 2.1 for this particular case. So, suppose that $\psi : X \to 2^E$ is l.s.c. and convex-valued, and $g : X \to E$ is a continuous map such that $\psi(x)$ is compact whenever $g(x) \notin \psi(x)$. Also, let $\varepsilon > 0$ and let $\mathscr{V}$ be an open and locally finite cover of E such that $\mathrm{diam}_d(V) < \varepsilon$ for every $V \in \mathscr{V}$. Since g is continuous, $\mathscr{U}_1 = \{g^{-1}(V) \cap \psi^{-1}(V) : V \in \mathscr{V}\}$ is a locally finite family of open subsets of X which refines $\{\psi^{-1}(V) : V \in \mathscr{V}\}$. Then, $A = X \setminus \bigcup \mathscr{U}_1$ is a closed subset of X, while $\psi \restriction A$ is compact-valued. Indeed, if $g(x) \in \psi(x)$, then $x \in \psi^{-1}(V)$ whenever $V \in \mathscr{V}$ and $g(x) \in V$. That is, $x \in A$ implies $g(x) \notin \psi(x)$, so, in this case, $\psi(x)$ must be compact. Thus, $\{\psi^{-1}(V) : V \in \mathscr{V}\}$ is an open (in X) and point-finite (in A) cover of A such that $|\mathscr{V}| \leq \tau$ because $\mathscr{V}$ is locally-finite and $w(E) \leq \tau$. Since X is τ-collectionwise normal, by [18, Lemma 1.6], $\{\psi^{-1}(V) : V \in \mathscr{V}\}$ has an open and locally finite (in X) refinement $\mathscr{U}_2$ which covers A. Then, $\mathscr{U} = \mathscr{U}_1 \cup \mathscr{U}_2$ is an open and locally finite cover of X which refines $\{\psi^{-1}(V) : V \in \mathscr{V}\}$. For every $U \in \mathscr{U}$ take a fixed $V_U \in \mathscr{V}$ such that $U \subset \psi^{-1}(V_U)$ and a point $e(U) \in V_U$ provided $V_U \neq \varnothing$. Next, take a partition of unity $\{\xi_U : U \in \mathscr{U}\}$ on X which is index subordinated to the cover $\mathscr{U}$, see [15]. Finally, define a continuous map $f : X \to E$ by $f(x) = \sum \{\xi_U(x) \cdot e(U) : U \in \mathscr{U}\}$, $x \in X$. This f is an ε-selection for ψ. $\square$

As far as the role of the family $\mathscr{C}'(E)$ is concerned, the arguments in the proof of Theorems 1.2 and 3.1 were based only on the property that if $\varphi : X \to \mathscr{C}'(E)$ and $g : X \to E$ is an ε-selection for φ for some $\varepsilon > 0$, then the set-valued mapping $\psi(x) = \overline{\varphi(x) \cap B_\varepsilon(g(x))}$, $x \in X$, is such that $\psi(x)$ is compact whenever $g(x) \notin \psi(x)$. That is, this resulting ψ is always as in Claim 2.1, and the inductive construction can be carried on.

Motivated by this, we shall say that a mapping $\psi : X \to \mathscr{F}(E)$ has a *selection* $\mathscr{C}(E)$-*deficiency* if there exists a continuous $g : X \to E$ such that $\psi(x) \in \mathscr{C}(E)$ for every $x \in X$ for which $g(x) \notin \psi(x)$. Clearly, every $\varphi : X \to \mathscr{C}'(E)$ has this property, for instance take $g : X \to E$ to be any constant map. However, there

are natural examples of mappings $\varphi : X \to \mathscr{F}(E)$ which have a selection $\mathscr{C}(E)$-deficiency and are not $\mathscr{C}'(E)$-valued, see next section. Related to this, we don't know if such mappings may have continuous selections in the case of collectionwise normal spaces.

Question 1. Let X be a collectionwise normal space, E be a Banach space, and let $\varphi : X \to \mathscr{F}(E)$ be an l.s.c. convex-valued mapping which has a selection $\mathscr{C}(E)$-deficiency. Then, is it true that φ has a continuous selection?

Another aspect of improving Theorem 1.1 is related to the range of the set-valued mapping. In this regard, Theorem 3.1 remains valid without any change in the arguments if the Banach space E is replaced by a closed convex subset Y of E. On the other hand, if Y is a completely metrizable absolute retract for the metrizable spaces, then for every collectionwise normal space X and closed $A \subset X$, every continuous map $g : A \to Y$ can be continuously extended to the whole of X, see, e.g., [19]. In particular, this is true for every convex G_δ-subset Y of a Banach space E. Namely, Y is an absolute retract for metrizable spaces being convex (by Dugundji's extension theorem [6]), and is also completely metrizable being a G_δ-subset of a complete metric space. Motivated by this and the relations between extensions and selections demonstrated in the proof of Theorem 1.2, we have also the following question.

Question 2. Let E be a Banach space, $Y \subset E$ be a convex G_δ-subset of E, X be a collectionwise normal space, and let $\varphi : X \to \mathscr{C}'(Y)$ be an l.s.c. convex-valued mapping. Then, is it true that φ has a continuous selection?

Question 2 is similar to Michael's G_δ-problem [17, Problem 396] if for a paracompact space X and a convex G_δ-subset Y of a Banach space, every l.s.c. convex-valued $\varphi : X \to \mathscr{F}(Y)$ has a continuous selection. In general, the answer to this latter problem is in the negative due to a counterexample constructed by Filippov [8, 9]. However, Michael's G_δ-problem was resolved in the affirmative in a number of partial cases. The solution in some of these cases remains valid for Question 2 as well. For instance, if Y is a countable intersection of open convex sets, then the closure convex-hull $\overline{\operatorname{conv}(K)}$ of every compact subset of Y will be still a subset of Y, see [17]. In this case, by a result of [2], φ will have an l.s.c. convex-valued selection $\psi : X \to \mathscr{C}(Y)$ (i.e., $\psi(x) \subset \varphi(x)$ for all $x \in X$). Hence, φ will have a continuous selection because, by Theorem 1.1, so does ψ. If the covering dimension of X is bounded (i.e., $\dim(X) < \infty$), then the answer to Question 2 is also "yes", this follows directly from a selection theorem in [10]. The answer to Question 2 is also "yes" if X is strongly countable-dimensional (i.e., a countable union of closed finite-dimensional subsets). In this case, there exists a metrizable (strongly) countable-dimensional space Z, a continuous map $g : X \to Z$ and an l.s.c. mapping $\psi : Z \to \mathscr{C}(Y)$ such that $\psi(g(x)) \subset \varphi(x)$ for every $x \in X$, see, for instance, the proof of [18, Theorem 5.3]. Next, define a mapping $\Phi : Z \to \mathscr{F}(Y)$

by $\Phi(z) = \overline{\text{conv}(\psi(z))}^Y$, $z \in Z$, where the closure is in Y. According to [16, Propositions 2.3 and 2.6], Φ remains l.s.c., and, by [11, Corollary 1.2], it admits a continuous selection $h : Z \to Y$. Then, $f = h \circ g$ is a continuous selection for φ because $\Phi(g(x)) \subset \varphi(x)$ for all $x \in X$.

4. Controlled Selections and Countable Paracompactness

A function $\xi : X \to \mathbb{R}$ is *lower (upper) semi-continuous* if the set

$$\{x \in X : \xi(x) > r\} \quad (\text{respectively}, \{x \in X : \xi(x) < r\})$$

is open in X for every $r \in \mathbb{R}$. If (E, d) is a metric space, $\varphi : X \to 2^E$ and $\eta : X \to (0, +\infty)$, then we shall say that $g : X \to E$ is an *η-selection* for φ if $d(g(x), \varphi(x)) < \eta(x)$ for every $x \in X$.

In this section, we first prove the following characterization of countably paracompact normal spaces.

Theorem 4.1. *For a T_1-space X, the following are equivalent*:
  (a) *X is countably paracompact and normal.*
  (b) *If E is a separable Banach space, $\varphi : X \to \mathscr{F}(E)$ is an l.s.c. convex-valued mapping, $\eta : X \to (0, +\infty)$ is lower semi-continuous and $g : X \to E$ is a continuous η-selection for φ, then φ has a continuous selection $f : X \to E$ such that $d(f(x), g(x)) < \eta(x)$ for all $x \in X$.*
  (c) *If $\varphi : X \to \mathscr{C}(\mathbb{R})$ is an l.s.c. convex-valued mapping, $\varepsilon > 0$ and $g : X \to \mathbb{R}$ is a continuous ε-selection for φ, then φ has a continuous selection $f : X \to \mathbb{R}$ such that $d(f(x), g(x)) < \varepsilon$ for all $x \in X$.*

Proof. (a) $\Rightarrow$ (b). Let X be a countably paracompact normal space, and let E, φ, η and g be as in (b). Since φ is l.s.c. and g is continuous, $\xi(x) = d(g(x), \varphi(x))$, $x \in X$, is an upper semi-continuous function such that $\xi(x) < \eta(x)$ for all $x \in X$ because g is an η-selection for φ. Since X is countably paracompact and normal, by a result of [3, 4, 14] (see, also, [7, 5.5.20]) there exists a continuous function $\alpha : X \to \mathbb{R}$ such that $\xi(x) < \alpha(x) < \eta(x)$ for every $x \in X$. Then, define an l.s.c. mapping $\psi : X \to \mathscr{F}(E)$ by $\psi(x) = \overline{\varphi(x) \cap B_{\alpha(x)}(g(x))}$, $x \in X$. Since ψ is convex-valued, by [16, Theorem 3.1″], ψ has a continuous selection $f : X \to E$. In particular, $d(f(x), g(x)) \leq \alpha(x) < \eta(x)$ for all $x \in X$.

Since (b) $\Rightarrow$ (c) is obvious, we complete the proof showing that (c) $\Rightarrow$ (a). So, suppose that (c) holds. If A and B are disjoint closed subsets of X, then $\varphi(x) = \{0\}$ if $x \in A$, $\varphi(x) = \{1\}$ if $x \in B$, and $\varphi(x) = [0, 1]$ otherwise, is an l.s.c. convex-valued mapping $\varphi : X \to \mathscr{C}(\mathbb{R})$. If $g(x) = \frac{1}{2}$, $x \in X$, then g is a continuous 1-selection for φ, and, by (c), φ has a continuous selection $f : X \to \mathbb{R}$. According to the definition of φ, we get that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$, hence X is normal. To show that X is countably paracompact, let $\{F_n : n < \omega\}$ be a decreasing

sequence of closed subsets of X such that $F_0 = X$ and $\bigcap\{F_n : n < \omega\} = \varnothing$. Next, for every $x \in X$, let $n(x) = \max\{n < \omega : x \in F_n\}$. Then, define a convex-valued mapping $\varphi : X \to \mathscr{C}(\mathbb{R})$ by $\varphi(x) = [0, 2^{-n(x)}]$, $x \in X$. Observe that φ is l.s.c. because $z \in X \setminus F_{n(x)+1}$ implies that $n(z) \leq n(x)$ and, therefore, $\varphi(x) = [0, 2^{-n(x)}] \subset [0, 2^{-n(z)}] = \varphi(z)$. Finally, observe that $g(x) = 1$, $x \in X$, is a continuous 1-selection for φ. Hence, by (c), φ has a continuous selection $f : X \to \mathbb{R}$ such that $|g(x) - f(x)| = 1 - f(x) < 1$ for every $x \in X$, or, in other words, $f(x) > 0$ for all $x \in X$. Finally, define $W_n = f^{-1}((-\infty, 2^{-n+1}))$, $n < \omega$. Thus, we get a sequence $\{W_n : n < \omega\}$ of open open subsets of X such that $F_n \subset W_n$ for every $n < \omega$. Indeed, $x \in F_n$ implies $n \leq n(x)$, so $f(x) \in \varphi(x) = [0, 2^{-n(x)}] \subset [0, 2^{-n}] \subset [0, 2^{-n+1}]$. Since $\overline{W_n} \subset f^{-1}((-\infty, 2^{-n+1}])$, $n < \omega$, and $f(x) > 0$ for every $x \in X$, we have that $\bigcap\{\overline{W_n} : n < \omega\} = \varnothing$. That is, X is countably paracompact, see [7, Theorem 5.2.1]. $\square$

For collectionwise normal spaces we have a very similar result which, in particular, illustrates the difference with countably paracompact ones (see, (c) of Theorem 4.1).

Proposition 4.2. *Let E be a Banach space, X be a collectionwise normal space, $\varphi : X \to \mathscr{C}'(E)$ be an l.s.c. convex-valued mapping, and let $g : X \to E$ be a continuous ε-selection for φ for some $\varepsilon > 0$. Then, φ has a continuous selection $f : X \to E$ such that $d(f(x), g(x)) \leq \varepsilon$ for every $x \in X$.*

Proof. Define a mapping $\psi : X \to \mathscr{F}(E)$ by $\psi(x) = \overline{B_\varepsilon(g(x))}$, $x \in X$. Then, ψ is convex-valued and d-proximal continuous in the sense of [12]. Define another mapping $\theta : X \to \mathscr{F}(E)$ by $\theta(x) = \overline{\varphi(x) \cap B_\varepsilon(g(x))}$, $x \in X$. According to [16, Propositions 2.3 and 2.5], θ is l.s.c. and clearly it is also convex-valued. Finally, observe that $\theta(x) \subset \psi(x)$ for every $x \in X$, while $\theta(x) \neq \psi(x)$ implies that $\theta(x)$ is compact. Then, by [13, Lemma 4.2], θ has a continuous selection $f : X \to E$. This f is as required. $\square$

Following the idea Proposition 4.2, one can extend Theorem 4.1 to the case of countably paracompact and τ-collectionwise normal spaces.

Theorem 4.3. *For a T_1-space X and an infinite cardinal number τ, the following are equivalent:*

(a) *X is countably paracompact and τ-collectionwise normal.*

(b) *If E is a Banach space with $w(E) \leq \tau$, $\varphi : X \to \mathscr{C}'(E)$ is l.s.c. and convex-valued, $\eta : X \to (0, +\infty)$ is lower semi-continuous, and $g : X \to Y$ is a continuous η-selection for φ, then φ has a continuous selection $f : X \to E$ such that $d(f(x), g(x)) < \eta(x)$ for all $x \in X$.*

Proof. (a) $\Rightarrow$ (b). As in (a) of the proof of Theorem 4.1, there exists a continuous function $\alpha : X \to (0, +\infty)$ such that $d(g(x), \varphi(x)) < \alpha(x) < \eta(x)$ for

every $x \in X$. Next, as in the proof of Proposition 4.2, define a (d-proximal) continuous $\psi(x) = \overline{B_{\alpha(x)}(g(x))}$, $x \in X$, and another l.s.c. $\theta : X \to \mathscr{F}(E)$ by $\theta(x) = \overline{\varphi(x) \cap B_{\alpha(x)}(g(x))}$, $x \in X$. As in the proof of [13, Lemma 4.2], this θ has the Selection Factorization Property in the sense of [18]. Hence, by [18, Proposition 4.3], θ has a continuous selection $f : X \to E$. According to the definition of θ, we get that $d(f(x), g(x)) < \eta(x)$ for all $x \in X$.

(b) $\Rightarrow$ (a). This implication is based on standard arguments. In fact, X will be countably paracompact and normal by Theorem 4.1. To show that X is also τ-collectionwise normal, let $\mathscr{D}$ be a discrete family of closed subsets of X, with $|\mathscr{D}| \leq \tau$, and let $\ell_1(\mathscr{D})$ be the Banach space of all functions $y : \mathscr{D} \to \mathbb{R}$, with $\sum \{|y(D)| : D \in \mathscr{D}\} < \infty$, equipped with the norm $\|y\| = \sum \{|y(D)| : D \in \mathscr{D}\}$. Also, let $\vartheta(D) = 0$, $D \in \mathscr{D}$, be the origin of $\ell_1(\mathscr{D})$. For every $D \in \mathscr{D}$, consider the function $e_D : \mathscr{D} \to \mathbb{R}$ defined by $e_D(D) = 1$ and $e_D(T) = 0$ for $T \in \mathscr{D} \setminus \{D\}$. Then, $e_D \in \ell_1(\mathscr{D})$, $D \in \mathscr{D}$, and $\|e_D - \vartheta\| = 1$ for every $D \in \mathscr{D}$. Finally, define an l.s.c. mapping $\varphi : X \to \mathscr{C}'(\ell_1(\mathscr{D}))$ by $\varphi(x) = \{e_D\}$ if $x \in D$ for some $D \in \mathscr{D}$ and $\varphi(x) = \ell_1(\mathscr{D})$ otherwise. Then, φ is convex-valued and $g(x) = \vartheta$, $x \in X$, is a continuous 2-selection for φ. Since $w(\ell_1(\mathscr{D})) \leq \tau$, by (b), φ has a continuous selection $f : X \to \ell_1(\mathscr{D})$. Then, $U_D = f^{-1}(B_1(e_D))$, $D \in \mathscr{D}$, is a pairwise disjoint family of open subsets of X such that $D \subset U_D$, $D \in \mathscr{D}$. Since X is normal, this implies that it is also τ-collectionwise normal. $\qquad\square$

Exactly the same arguments as those in the proof of Theorem 4.3 show that if X is τ-collectionwise normal, E is a Banach space with $w(E) \leq \tau$, $\varphi : X \to \mathscr{C}'(E)$ is an l.s.c. convex-valued mapping, $\eta : X \to (0, +\infty)$ is continuous, and $g : X \to E$ is a continuous η-selection, then φ has a continuous selection $f : X \to E$ such that $d(f(x), g(x)) \leq \eta(x)$ for all $x \in X$. Motivated by this and Proposition 4.2, we have the following natural question.

Question 3. Let X be a collectionwise normal space, E be a Banach space, $\varphi : X \to \mathscr{C}'(E)$ be an l.s.c. convex-valued mapping, and let $g : X \to E$ be a continuous η-selection for φ for some lower semi-continuous function $\eta : X \to (0, +\infty)$. Then, does φ have a continuous selection $f : X \to E$ with $d(f(x), g(x)) \leq \eta(x)$ for all $x \in X$?

Let us point out that the answer to Question 3 is "yes" if so is the answer to Question 1. Indeed, if $\eta : X \to (0, +\infty)$ is lower semi-continuous and $g : X \to E$ is continuous, then the mapping $\psi(x) = B_{\eta(x)}(g(x))$, $x \in X$, will have an open graph. If g is also an η-selection for $\varphi : X \to \mathscr{C}'(E)$, then $\theta(x) = \overline{\varphi(x) \cap \psi(x)}$, $x \in X$, will have a selection $\mathscr{C}(E)$-deficiency. Finally, if $f : X \to E$ is a continuous selection for θ, then $d(f(x), g(x)) \leq \eta(x)$ for all $x \in X$.

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